UDC 531.36

ALGEBRAIC CRITERION FOR THE STOCHASTIC STABILITY OF LINEAR SYSTEMS WITH THE PARAMETRIC ACTION OF CORRELATED WHITE NOISE

PMM Vol. 36, №3, 1972, pp. 546-551

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(Received February 5, 1971)

We give the necessary and sufficient conditions for the exponential mean square stability of linear systems with constant coefficients subjected to the action of correlated white noise. These conditions are expressed in terms of the transfer functions. We present example.

Papers [1 – 9] were devoted to the stability problem for stochastic systems. Exponential mean square stability of linear systems with white noise was examined in detail in [1, 4, 6 – 9]. The stability criterion proposed in [3, 4] requires the computations of determinants of upto order $\nu(\nu+1)/2$, where ν is the system's order. A criterion was established in [9] for a special class of systems, requiring a knowledge only of the system's transfer matrix from noise to outputs, moreover, taking count in the systems of the number of perturbing noise sometimes makes it possible to avoid the laborious calculations of higher-order determinants. In this paper the investigative method in [9] is extended to a wider class of linear systems which are under the parametric action of dependent noise. In many cases the criterion proposed here permits us to restrict ourselves to the computation of determinants of orders less than both the number of noise perturbing the system as well as the quantity $\nu(\nu+1)/2$. Just as in [9] the criterion is applicable to systems given by a transfer matrix from noise to outputs. Everywhere below, by the stability of a system with noise we mean the exponential mean square stability.

1. The class of systems being considered. We reckon that the whole collection of noise in the system can be divided into n groups such that the noise from different groups are independent of each other. By the symbol k_l we denote the number of noise in the group numbered l, by the symbol $W_l^{\cdot}$, a vector comprised from the k_l noise of this group, by the symbol B_{ll}, the $(k_l \times k_l)$-dimensional covariance matrix of the vector-valued noise $W_l^{\cdot}$

$$M[W_l^{\cdot}(t)][W_l^{\cdot}(s)]' = B_{ll}\delta(t-s)$$

The prime denotes the matrix transpose. We consider a system of linear Ito differential equations

$$x^{\cdot} = Px + \sum_{l=1}^{n} q_l \sigma_l' W_l^{\cdot}, \qquad \sigma_l = r_l' x \tag{1.1}$$

Here P, q_l, r_l are constant matrices. The vector x of the variables is of dimension ν, the matrix P is of dimension $\nu \times \nu$.

We shall examine two types of systems. In the first type belong systems (1.1) in which

q_l are ν-dimensional vectors and r_l are of dimension $\nu \times k_l$. In systems of the first type σ_l are k_l-dimensional column vectors, and $\sigma_l' W_l^{\boldsymbol{\cdot}}$ are scalar noise. In the second type belong systems (1.1) in which q_l are of dimension $\nu \times k_l$ and r_l are k_l-dimensional column vectors. In systems of the second type σ_l are scalars, and the prime in the first formula in (1.1) can be dropped. We see that the linear systems considered in [9] belong both to the first type as well as to the second. We take it that system (1.1) is completely controllable and observable.

We introduce the matrices

$$\chi_{lm}(\lambda) = r_l' (P - \lambda I)^{-1} q_m, \qquad (l, m = 1, 2, \ldots, n) \tag{1.2}$$

where λ is a complex number. As is well known, system (1.1) can be specified by its transfer matrix to within a transformation of the form $y = Sx$, $\det S \neq 0$. The transfer matrix of system (1.1) from the inputs $\varphi_m = \sigma_m' W_m$ $(m = 1, 2, \ldots, n)$ to the outputs σ_l $(l = 1, 2, \ldots, n)$ consists of blocks χ_{lm} and has the form

$$\chi(\lambda) = \begin{Vmatrix} \chi_{11}(\lambda) \ldots \chi_{1n}(\lambda) \\ \vdots \qquad \vdots \\ \chi_{n1}(\lambda) \ldots \chi_{nn}(\lambda) \end{Vmatrix} \tag{1.3}$$

The fundamental characteristics of the matrix dimensions $(k = k_1 + k_2 + \ldots + k_n)$ for systems of the first and second types are presented below

	$W_l^{\boldsymbol{\cdot}}$	q_l	r_l	σ_l	$\sigma_l W_l^{\boldsymbol{\cdot}}$	$\chi_{lm}(\lambda)$	$\chi(\lambda)$
1	$k_l \times 1$	$\nu \times 1$	$\nu \times k_l$	$k_l \times 1$	scalar	$k_l \times 1$	$k \times n$
2	$k_l \times 1$	$\nu \times k_l$	$\nu \times 1$	scalar	$k_l \times 1$	$1 \times k_l$	$n \times k$

We represent the matrices $\chi_{lm}(\lambda)$ in the form

$$\chi_{lm}(\lambda) = \frac{\gamma_{lm}(\lambda)}{\Delta_{lm}(\lambda)} \tag{1.4}$$

where $\gamma_{lm}(\lambda)$ is a matrix polynomial while $\Delta_{lm}(\lambda)$ is a scalar polynomial. Here the degree of the matrix polynomial $\gamma_{lm}(\lambda)$ is less than the degree of the scalar polynomial $\Delta_{lm}(\lambda)$ whose leading coefficient is taken to be unity.

2. Stability conditions. Consider a system of first type. We set

$$\delta_{lm}(\lambda) = [\gamma_{lm}(-\lambda)]' B_{ll} \gamma_{lm}(\lambda) \tag{2.1}$$

we define a scalar polynomial $\tau_{lm}(\lambda)$ by the equation

$$\tau_{lm}(\lambda) \Delta_{lm}(-\lambda) + \tau_{lm}(-\lambda) \Delta_{lm}(\lambda) = \delta_{lm}(\lambda) \tag{2.2}$$

under the condition that the degree of $\tau_{lm}(\lambda)$ is less than the degree of the polynomial $\Delta_{lm}(\lambda)$. In the system of first type we set up an $(n \times n)$-dimensional matrix $R = \| \rho_{lm} \|$ by the formulas

$$\rho_{lm} = \lim_{|\lambda| \to \infty} \frac{\tau_{lm}(\lambda)}{\Delta_{lm}(\lambda)} \tag{2.3}$$

In a system of second type we set up the matrix R by the relations

$$\delta_{lm}(\lambda) = \gamma_{lm}(\lambda) B_{mm} [\gamma_{lm}(-\lambda)]' \tag{2.4}$$

and by formulas (2.2), (2.3).

Theorem 1. Let system (1.1) be a system of first or second type, and let the vector-valued noise $W_1^{\boldsymbol{\cdot}}, \ldots, W_n^{\boldsymbol{\cdot}}$ be mutually independent. For the stability of the system it is necessary and sufficient that its transfer matrix (1.3) have poles in the open left halfplane and that the eigenvalues of the($n \times n$)-dimensional matrix $R = \| \rho_{lm} \|$, defined by formulas (2.1)—(2.3) or (2.2) – (2.4), be less than unity in absolute value.

The proof of Theorem 1 is carried out by the same scheme as for Theorem 1 in [9]. Here we should take into account that the stochastic derivative of the quadratic form $V(x) = x'Hx$ relative to a system of the first type reduces to the form

$$LV = 2x'HPx + x' \left(\sum_{i=1}^{n} q_i' H q_i r_i B_{ii} r_i' \right) x$$

while for a system of the second type, to the form

$$LV = 2x'HPx + x' \left(\sum_{i=1}^{n} \mathrm{sp}\, [q_i' H q_i B_{ii}^{\cdot}]\, r_i r_i' \right) x$$

The symbol sp denotes the trace (spur) of the matrix.

Note 1. To determine the matrix R it is necessary to solve the following problem. Suppose we are given

$$\Delta(\lambda) = \lambda^s + \Delta_1 \lambda^{s-1} + \ldots + \Delta_s, \qquad \delta(\lambda) = \sum_{i=1}^{s} \delta_i \lambda^{2(s-i)}$$

where $\Delta(\lambda)$ is a Hurwitz polynomial. We are required to compute the quantity

$$\rho = \lim_{|\lambda| \to \infty} \lambda \frac{\tau(\lambda)}{\Delta(\lambda)} \qquad (\tau(\lambda) = \tau_1 \lambda^{s-1} + \ldots + \tau_s) \tag{2.5}$$

where the polynomial $\tau(\lambda)$ is determined from the equation

$$\tau(\lambda)\, \Delta(-\lambda) + \tau(-\lambda)\, \Delta(\lambda) = \delta(\lambda) \tag{2.6}$$

By equating the coefficients of like even powers of λ in the left and right hand sides of (2.6), we obtain a linear algebraic system in the s unknown coefficients of the polynomial $2\tau(-\lambda) = \beta_1 \lambda^{s-1} + \ldots + \beta_s$. Since $\Delta(\lambda)$ is a Hurwitz polynomial, the determinant of this system is positive and, consequently, Eq. (2.2) has a unique solution for any right hand side. The quantity ρ is computed from the formula

$$\rho = (-1)^{s-1} \beta_1 / 2 \tag{2.7}$$

Note 2. It can be shown that if $\Delta(\lambda)$ is a Hurwitz polynomial in (2.6), while the polynomial $\delta(\lambda)$ is nonnegative for $\lambda = i\omega$ $(-\infty < \omega < +\infty)$, then the quantity (2.3) is nonnegative. Because of this, R is a matrix with nonnegative elements.

Note 3. The ($n \times n$)-matrix R with nonnegative elements has eigenvalues less than unity in absolute value if and only if all the successive principal minors of the matrix $(I - R)$ are positive [10]. Let the matrix R be nonzero. By μ_0 we denote the smallest real positive root of the equation

$$\det (I - \mu R) = 0 \tag{2.8}$$

Then the spectrum of the matrix μR $(\mu > 0)$ lies inside the unit circle for all $\mu < \mu_0$ and only for these values of μ.

Example. Consider the system

$$\begin{aligned} y^{(2)} + (1 + \psi_1^{\cdot})\, y^{(1)} + (3 + \psi_2^{\cdot})\, y + (\psi_2^{\cdot} - 1)\, z = 0 \\ z^{(2)} + (1 + \zeta_1^{\cdot})\, y^{(1)} + (1 + \zeta_2^{\cdot})\, y = 0 \end{aligned} \tag{2.9}$$

Here $\psi_1^{\cdot}, \psi_2^{\cdot}, \zeta_1^{\cdot}, \zeta_2^{\cdot}$ is the white noise. We assume that the noise $\psi_i^{\cdot}$ is uncorrelated with the noise $\zeta_j^{\cdot}$. We denote

$$W_1^{\cdot} = \begin{Vmatrix} \psi_1^{\cdot} \\ \psi_2^{\cdot} \end{Vmatrix}, \qquad W_2^{\cdot} = \begin{Vmatrix} \zeta_1^{\cdot} \\ \zeta_2^{\cdot} \end{Vmatrix}$$

We assume that the covariance matrices of noise $W_1^{\cdot}$, $W_2^{\cdot}$ equal, respectively,

$$B_{11} = \| b_{ij} \| = \begin{Vmatrix} 0.36 & -0.18 \\ -0.18 & 0.09 \end{Vmatrix}, \qquad B_{22} = \| c_{ij} \| = \begin{Vmatrix} 0.12 & 0.13 \\ 0.13 & 0.18 \end{Vmatrix}$$

We find the transfer matrix from the inputs

$$\varphi_1 = \psi_1^{\cdot} y^{(1)} + \psi_2^{\cdot} (y + z), \qquad \varphi_2 = \zeta_1^{\cdot} y^{(1)} + \zeta_2^{\cdot} y$$

to the outputs

$$\sigma_1 = \begin{Vmatrix} y^{(1)} \\ y + z \end{Vmatrix}, \qquad \sigma_2 = \begin{Vmatrix} y^{(1)} \\ y \end{Vmatrix}$$

We obtain

$$\sigma_1 = -\frac{\gamma_{11}\varphi_1 + \gamma_{12}\varphi_2}{\Delta(\lambda)}, \qquad \sigma_2 - = \frac{\gamma_{21}\varphi_1 + \gamma_{22}\varphi_2}{\Delta(\lambda)},$$

$$\gamma_{11}(\lambda) = \begin{Vmatrix} \lambda^3 \\ \lambda^2 - \lambda - 1 \end{Vmatrix}, \quad \gamma_{12}(\lambda(= \begin{Vmatrix} \lambda \\ \lambda^2 + \lambda + 4 \end{Vmatrix}, \quad \gamma_{21}(\lambda) = \begin{Vmatrix} \lambda^3 \\ \lambda^2 \end{Vmatrix}, \quad \gamma_{22}(\lambda) = \begin{Vmatrix} \lambda \\ 1 \end{Vmatrix}$$

$$\Delta(\lambda) = \lambda^4 + \lambda^3 + 3\lambda^2 + \lambda + 1$$

Clearly, $\Delta(\lambda)$ is a Hurwitz polynomial. The polynomials

$$\delta_{lm}(\lambda) = [\gamma_{lm}(-\lambda)]' B_{ll}\gamma_{lm}(\lambda) \qquad (l, m = 1, 2)$$

take the form

$$\begin{aligned} \delta_{11}(\lambda) &= -b_{11}\lambda^6 + (2b_{12} + b_{22})\lambda^4 - 3b_{22}\lambda^2 + b_{22} \\ \delta_{12}(\lambda) &= b_{22}\lambda^4 + (-b_{11} - 2b_{12} + 7b_{22})\lambda^2 + 16b_{22} \\ \delta_{21}(\lambda) &= (-c_{11})\lambda^6 + c_{22}\lambda^4 \\ \delta_{22}(\lambda) &= (-c_{11})\lambda^2 + c_{22} \end{aligned}$$

We determine the quantity (2.7). We obtain

$$\rho = (\delta_4 - \delta_1) + (\delta_2 - \delta_3) / 2$$

Consequently, the elements of the (2×2)-dimensional matrix $R = \| \rho_{ij} \|$ are the numbers

$$\begin{aligned} \rho_{11} &= b_{11} + b_{12} + 3b_{22} = 0.45, \quad \rho_{12} = b_{11} / 2 + b_{12} + 13\, b_{22} = 1.17 \\ \rho_{21} &= c_{11} + c_{22} / 2 = 0.21, \quad \rho_{22} = c_{11} / 2 + c_{22} = 0.24 \end{aligned}$$

For (2×2)-dimensional matrices R the smallest real root of (2.8) is given by the formula

$$\mu_0 = \begin{cases} (2 \det R)^{-1} [\operatorname{sp} R - \sqrt{(\operatorname{sp} R)^2 - 4 \det R}], & \det R \neq 0 \\ (\operatorname{sp} R)^{-1} & \det R = 0 \end{cases}$$

In the case being considered $\mu_0 \approx 1.17 > 1$. Thus, system (2.9) is stable. Stability is preserved if the intensities of the noise increases by less than $\sqrt{\mu_0} \approx 1.08$ times. Otherwise the system loses stability. (It is assumed that the correlation coefficients remain the same as the noise intensity varies).

3. Linear differential equations of order ν. Let us consider the oft-encountered case when a system of the first type is described by a linear differential equation of order ν

$$y^{(\nu)} + (\Delta_1 + \eta_\nu^\cdot)\, y^{(\nu-1)} + \ldots + (\Delta_\nu + \eta_1^\cdot)\, y = 0 \tag{3.1}$$

Here the noise $\eta_i^\cdot$ comprises one group from the ν noise $W^\cdot$ with covariance matrix B

$$MW^\cdot(t)\,[W^\cdot(s)]' = B\delta(t-s) \qquad (W^\cdot = \| \eta_i^\cdot \|_{i=1}^{\nu}, \qquad B = \| b_{ij} \|)$$

Consider the polynomial

$$\delta(\lambda) = \sum_{i,\, j=1}^{\nu} (-1)^{i-1} b_{ij} \lambda^{i+j-2} \tag{3.2}$$

From Theorem 1 we conclude the following.

Theorem 2. Equation (3.1) is stable if and only if $\Delta(\lambda) = \lambda^\nu + \Delta_1\lambda^{\nu-1} + \ldots + \Delta_\nu$ is a Hurwitz polynomial and the nonnegative number ρ defined by formulas (2.5), (2.6) (3.2) is less than unity.

Equation (3.1) was considered in [3, 4, 7]. The stability criterion for (3.1) proposed in [4, 7] is equivalent to Theorem 2. Its theoretical foundation is ideologically close to the method of the present paper; the difference in the criteria is in the method of computing certain characteristics of Eq. (3.1). As was noted in [7], only those correlation coefficients b_{ij} for which the number $i+j$ is even, prove to have an influence on the stability of Eq. (3.1). This follows from the fact that in polynomial (3.2) only the coefficients of even powers of λ are nonzero, and they are a linear combination of correlation coefficients with an even sum of the indices.

Stability conditions for second- and third-order Eqs. (3.1) can be found in [4]. We present the stability conditions for differential equations (3.1) for $\nu = 4$ and 5, obtained with the aid of Theorem 2. It is assumed that the trivial solution of the equations being considered is asymptotically stable in the absence of the noise . For the equation

$$y^{(4)} + (a + \eta_4^\cdot)\, y^{(3)} + (b + \eta_3^\cdot)\, y^{(2)} + (c + \eta_2^\cdot)\, y^{(1)} + (d + \eta_1^\cdot)\, y = 0$$

the stability condition is

$$(ab - c)\, b_{11} + d\,[a(-2b_{13} + b_{22}) + c(-2b_{24} + b_{33}) + (bc - ad)\, b_{44}] < 2d\,(abc - c^2 - a^2 d)$$

for the equation

$$y^{(5)} + (a + \eta_5^\cdot)\, y^{(4)} + (b + \eta_4^\cdot)\, y^{(3)} + (c + \eta_3^\cdot)\, y^{(2)} + (d + \eta_2^\cdot)\, y^{(1)} + (e + \eta_1^\cdot) y = 0$$

the stability condition is

$$(cm - al)\, b_{11} + m(-2b_{13} + b_{22}) + l(2b_{15} - 2b_{24} + b_{33}) + k(-2b_{35} + b_{44}) + (bk - dl)\, b_{55} < 2mk - l^2 \quad (k = cd - be,\ l = ad - e,\ m = ab - c)$$

4. One result of the application of Theorem 1. We consider the linear system

$$x^\cdot = (P + \eta^\cdot)\, x, \qquad \eta^\cdot = \| \eta_{ij}^\cdot \|_{i,\, j=1}^{\nu} \tag{4.1}$$

where $\eta^\cdot$ is a matrix of dependent white noise $\eta_{ij}^\cdot$. We introduce the ν-dimensional column vectors

$$W_l^\cdot = \| \eta_{lj}^\cdot \|_{j=1}^{\nu}, \qquad MW_l^\cdot(t)\,[W_l^\cdot(s)]' = B_{ll}\delta(t-s)$$

$$U_l^\cdot = \| \eta_{il}^\cdot \|_{i=1}^{\nu}, \qquad MU_l^\cdot(t)\,[U_l^\cdot(s)]' = C_{ll}\delta(t-s)$$

System (4.1) can be written as

$$x^{\cdot} = Px + \sum_{m=1}^{\nu} e_m x' W_m^{\cdot} \tag{4.2}$$

or else as

$$x^{\cdot} = Px + \sum_{m=1}^{\nu} x' e_m U_m^{\cdot} \tag{4.3}$$

Here e_l is a vector of dimension ν, in which the l-th component equals unity and the remaining components are zero.

If in system (4.2) the noise $W_1^{\cdot}, \ldots, W_\nu^{\cdot}$ is independent, then (4.2) is a system of the first type, however, if in system (4.3) the noise $U_1^{\cdot}, \ldots, U_\nu^{\cdot}$ is independent, then system (4.3) is a system of the second type. We introduce the matrix polynomial $D(\lambda) = (\lambda I - P)^{-1} \det(\lambda I - P)$. We determine the ($\nu \times \nu$)-dimensional matrix R by formulas (2.2), (2.3). Here, in (2.2) we take it that

$$\delta_{lm}(\lambda) = e_m' [D(-\lambda)]' B_{ll} D(\lambda) e_m \tag{4.4}$$

if (4.2) is a system of the first type, or

$$\delta_{lm}(\lambda) = e_l' D(\lambda) C_{mm} [D(-\lambda)]' e_l \tag{4.5}$$

if (4.3) is a system of the second type. We obtain the following assertion by applying Theorem 1.

Corollary 1. In system (4.1) let P be a Hurwitz matrix, while in the noise matrix $\eta^{\cdot}$ either the rows $(W_l^{\cdot\prime})$ or the columns $(U_l^{\cdot})$ are mutually independent. Let matrix R be determined by formulas (2.2), (2.3), (4.4) if the rows are independent, or by formulas (2.2), (2.3), (4.5) if the columns are independent. System (4.1) is stable if and only if the eigenvalues of R are less than unity in absolute value.

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Translated by N. H. C.

UDC 531.01

ON A PHASE AUTOMATIC FREQUENCY CONTROL EQUATION WITH A LAG AND A RECTANGULAR PHASE DETECTOR CHARACTERISTIC

PMM Vol. 36, №3, 1972, pp. 551-555
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A second-order piecewise-linear dynamic system with jumps in the representative point on the juncture lines is investigated on a cylindrical phase space.

We consider the equation

$$\frac{d^2\varphi}{dt^2} + h\,[1 - bF'(\varphi)]\frac{d\varphi}{dt} + F(\varphi) = \gamma \qquad \begin{pmatrix} b > 0, \ h > 0 \\ 0 \leqslant \gamma < 1 \end{pmatrix}$$

$$F(\varphi + 2k\pi) \equiv F(\varphi) \quad (k = 0, \pm 1, \ldots), \quad F(\varphi) = \begin{cases} -1 & \text{for} \quad -\pi < \varphi < 0 \\ 1 & \text{for} \quad 0 < \varphi < \pi \end{cases}$$

This equation describes the dynamics of a phase automatic frequency control (afc) system with an integrating filter [1, 2] and a rectangular phase detector characteristic [3] with an approximate accounting for the lag [1]. It has no meaning for values of φ at which $F(\varphi)$ suffers discontinuities. By introducing new variables and notation

$$t^\circ = ht, \quad y = \frac{1}{h}\frac{d\varphi}{dt}, \quad \alpha = \frac{h^2\pi}{(1+\gamma)}, \quad \beta = \frac{h^2\pi}{(1-\gamma)} \geqslant \alpha$$

we replace the equation in the strips $-\pi < \varphi < 0$ and $0 < \varphi < \pi$ by the systems

$$\varphi^\cdot = y, \qquad y^\cdot = \alpha^{-1}\pi - y \quad (-\pi < \varphi < 0) \tag{1}$$

$$\varphi^\cdot = y, \qquad y^\cdot = -\beta^{-1}\pi - y \quad (0 < \varphi < \pi) \tag{2}$$

Here the dots denote differentiation with respect to t° ; a cylinder serves as the phase space of the system. Systems (1) and (2) permit us to trace the motion of the representative point upto the instant when it hits onto one of the straight lines $\varphi = 0$ or $\varphi = \pi$. The subsequent motion of the representative point requires an extension of the definition. We should indicate how much time it spends on the straight line, how it moves along it, at which point it leaves, and which of systems (1) or (2) describes its subsequent motion. We make use of the extended definition given in [4]. (When applying the formula (*) in [4] it is necessary to take into account the scales of t and y).

*) In [4] (English Version), page 756, line four from the top the erroneous equation $r = 2b$, $h > 0$, as given in the Russian Original Edition, should read $r = 2bh > 0$.